

Factorization in Positive Semirings

The Bi-UF Positive Conjecture and a Related Bi-LF Result

Omar Graia

Summer Workshop for Intrepid Mathematicians

SWIM 2026

- ① Preliminaries and Background
- ② The Bi-UF Positive Problem
- ③ The Related Bi-LF Question

General Notation

- $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$.
- $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the integers, rationals, and reals.
- $\mathbb{P}$ denotes the set of ordinary prime numbers.
- Unless stated otherwise, all monoids are **commutative** and **cancellative**.

The prototypical positive semiring in this talk is

$$\mathbb{N}_0.$$

Definition

A **monoid** $(M, *)$ is a nonempty set with an associative binary operation $*$ and an identity element e .

Examples.

- 1 $(\mathbb{N}_0, +)$ is a monoid with identity 0.
- 2 $(\mathbb{N}, \cdot)$ is a monoid with identity 1.
- 3 Square matrices form a monoid under multiplication, but this monoid is generally not commutative.

The first two examples will become the additive and multiplicative sides of $\mathbb{N}_0$.

Commutativity and Cancellativity

Let $(M, *)$ be a monoid.

- M is **commutative** if $a * b = b * a$ for all $a, b \in M$.
- M is **cancellative** if

$$a * c = b * c \implies a = b.$$

Examples.

- 1 $(\mathbb{N}_0, +)$ and $(\mathbb{N}, \cdot)$ are commutative and cancellative.
- 2 Matrix multiplication is generally neither commutative nor cancellative.

Groups and Group Completion

A **group** is a monoid in which every element has an inverse. A commutative group is called an **abelian group**.

For a cancellative monoid M , the **Grothendieck group** $\text{gp}(M)$ is the smallest group containing M .

Examples.

- ① $\text{gp}(\mathbb{N}_0, +) = \mathbb{Z}$.
- ② The group completion of $(\mathbb{N}, \cdot)$ is $(\mathbb{Q}_{>0}, \cdot)$.
- ③ If $M = \langle 2, 3 \rangle \subseteq \mathbb{N}_0$, then $\text{gp}(M) = \mathbb{Z}$.

Group completion will later let us subtract additive factorization relations.

Units, Associates, and Atoms

Let M be a monoid.

- A **unit** is an element with an inverse in M .
- Elements $a, b \in M$ are **associates** if $a = ub$ for some unit u .
- A nonunit $a \in M$ is an **atom** if $a = b * c$ forces b or c to be a unit.
- The set of atoms is denoted by $\mathcal{A}(M)$.

Examples.

- 1 $\mathcal{A}(\mathbb{N}_0, +) = \{1\}$.
- 2 $\mathcal{A}(\mathbb{N}, \cdot) = \mathbb{P}$.
- 3 In $(\{0\} \cup \mathbb{N}_{\geq 2}, +)$, the atoms are 2 and 3.

Definition

A monoid M is **atomic** if every nonunit is a finite product of atoms.

Examples.

- ① $(\mathbb{N}, \cdot)$ is atomic by the Fundamental Theorem of Arithmetic.
- ② $(\{0\} \cup \mathbb{N}_{\geq 2}, +)$ is atomic: every element is a sum of 2's and at most one 3.
- ③ The additive monoid of $\mathbb{N}_0[1/2]$ is not atomic, because every $x > 0$ satisfies

$$x = \frac{x}{2} + \frac{x}{2}.$$

Thus it has no positive additive atoms.

Prime Elements

In a monoid written multiplicatively, a nonunit p is **prime** if

$$p \mid xy \implies p \mid x \text{ or } p \mid y.$$

Fact

In a UFM, every atom is prime.

An atom need not be prime in a general monoid. In the additive monoid $\langle 2, 3 \rangle$,

$$2 \mid (3 + 3) = 6, \quad 2 \nmid 3.$$

Hence the atom 2 is not prime.

Factorizations and Lengths

Let M be atomic.

- A **factorization** of x is an expression of x as a product of atoms.
- Its **length** is the number of atoms appearing, counting repetitions.

In the additive monoid $\langle 2, 3 \rangle$,

$$6 = 3 + 3 = 2 + 2 + 2.$$

The two factorizations have lengths 2 and 3.

Factorization Properties

Let M be an atomic monoid.

- M is a **UFM** if every element has exactly one factorization, up to order and associates.
- M is an **HFM** if all factorizations of the same element have the same length.
- M is an **LFM** if distinct factorizations of the same element have distinct lengths.

For a semiring, the prefix **bi-** means that the property holds for both

$$(S, +) \quad \text{and} \quad S^\bullet = (S \setminus \{0\}, \cdot).$$

In particular, **bi-atomic** means that both monoids are atomic.

How the Properties Compare

Theorem (A–C–H–Z; G–Z)

For an atomic cancellative commutative monoid,

$$\text{UFM} \iff \text{HFM and LFM}.$$

Thus a UFM is both half-factorial and length-factorial.

The monoid $\langle 2, 3 \rangle$ is not a UFM and not an HFM, since

$$6 = 3 + 3 = 2 + 2 + 2,$$

but it is an LFM: two factorizations with the same value and the same length must be identical.

Positive Semirings

A **semiring** has addition and multiplication, identities 0 and 1, and distributivity.

A **positive semiring** is a subsemiring

$$S \subseteq \mathbb{R}_{\geq 0}$$

containing 1.

Examples.

- $\mathbb{N}_0$ has one additive atom, while its multiplicative atoms are the ordinary primes.
- In $\mathbb{N}_0[1/2]$, every positive element splits additively, but

$$\mathcal{U}(\mathbb{N}_0[1/2]) = \{2^k : k \in \mathbb{Z}\}.$$

- In $\mathbb{N}_0[\sqrt{2}]$, the additive monoid is freely generated by 1 and $\sqrt{2}$.

Bi-Factorization Properties

A positive semiring S is called

- a **bi-UFS** if $(S, +)$ and $S^\bullet$ are UFM's;
- a **bi-HFS** if $(S, +)$ and $S^\bullet$ are HFM's;
- a **bi-LFS** if $(S, +)$ and $S^\bullet$ are LFM's.

The prototype is $\mathbb{N}_0$:

- additively, $n = \underbrace{1 + \cdots + 1}_{n \text{ copies}}$;
- multiplicatively, every positive integer factors uniquely into primes.

Hence $\mathbb{N}_0$ is a bi-UFS, and therefore also a bi-HFS and a bi-LFS.

The Bi-UF Positive Conjecture

Baeth, Chapman, and Gotti introduced bi-atomic classes of positive semirings in 2021.

Conjecture (Baeth–Chapman–Gotti, 2021)

Is $\mathbb{N}_0$ the only positive semiring that is a bi-UFS?

They also asked the corresponding questions for bi-HFS and bi-LFS positive semirings.

- The bi-HF analogue is false: Gonzalez–Polo–Rodriguez constructed positive bi-HFS examples distinct from $\mathbb{N}_0$.
- The bi-LF question will be discussed after the Bi-UF theorem.

Recent Progress: Rational Settings

- 2026 L. Abedi, A. Arora, O. Chen, M. Gotti, E. Whitford, and J. Wu,
On the additive monoids of finitely generated rational semirings.
They established the conjecture for finitely generated rational positive semirings using information from the additive monoid.
- 2026 A. Deng, F. Gotti, and J. Zeng,
Factorizations in rational monogenic semidomains.
They established the conjecture for rational monogenic semidomains.

Recent Progress: Algebraic Settings

- 2026 F. Gotti, O. Graia, D. Han, and H. Liang,
The Bi-UF Positive Conjecture for quadratic monogenic semirings and related progress.
This was the first progress beyond the rational setting.
- 2026 A. Bilakanti, M. Gotti, A. Kandasamy, H. Liang, J. Liu, H. Polo, J. Yang, and A. Yao,
The Bi-UFS Positive Conjecture for algebraic semidomains.
They proved the monogenic algebraic case and the case generated by finitely many positive algebraic numbers.

Main Result and Proof Architecture

Theorem (Graia, 2026)

A positive semiring S is a bi-UFS if and only if

$$S = \mathbb{N}_0.$$

For the nontrivial direction, assume that S is a positive bi-UFS. The proof follows the chain

$$S \neq \mathbb{N}_0 \implies \exists \text{ irrational } a \in \mathcal{A}_+(S) \cap \mathcal{A}_\times(S) \implies \text{contradiction.}$$

The two main ingredients are additive coordinates and multiplicative unique factorization.

Additive Factorizations as Coordinates

When $(S, +)$ is a UFM, every element has a unique additive factorization. We write

$$Z_+(x) = \text{the additive factorization of } x, \quad \ell_+(x) = |Z_+(x)|.$$

For example, if

$$Z_+(x) = \{\{a, a, b\}\},$$

then $\ell_+(x) = 3$.

The additive atoms form a basis for $\text{gp}(S, +)$, and S is the nonnegative coordinate cone. Consequently, two additive expressions for the same element must contain the same multiset of atoms.

A Length Inequality

Lemma

If $(S, +)$ is a UFM, then for all nonzero $x, y \in S$,

$$\ell_+(xy) \geq \ell_+(x)\ell_+(y).$$

If

$$x = a_1 + \cdots + a_m, \quad y = b_1 + \cdots + b_n$$

are additive factorizations, then distributivity gives

$$xy = \sum_{i,j} a_i b_j.$$

Each product $a_i b_j$ is nonzero, so it contributes at least one additive atom.

Proposition (Baeth–Chapman–Gotti)

If S is a bi-HFS positive semiring, then

$$S \cap \mathbb{Q} = \mathbb{N}_0.$$

Since every UFM is an HFM, the proposition applies to every positive bi-UFS. Hence

- every element of $S \setminus \mathbb{N}_0$ is irrational;
- the only rational additive atom is 1.

For instance, $1/2 \in \mathbb{N}_0[1/2] \cap \mathbb{Q} \setminus \mathbb{N}_0$, so $\mathbb{N}_0[1/2]$ cannot be a bi-HFS.

Multiplicative Refinement

Lemma (G–Z)

If M is a UFM and $PQ = RT$, then there exist $u, v, w, x \in M$ such that

$$P = uv, \quad Q = wx, \quad R = uw, \quad T = vx.$$

Example in $(\mathbb{N}, \cdot)$.

$$6 \cdot 35 = 10 \cdot 21$$

refines as

$$6 = 2 \cdot 3, \quad 35 = 5 \cdot 7, \quad 10 = 2 \cdot 5, \quad 21 = 3 \cdot 7.$$

Unique factorization matches the prime factors across the equality.

Theorem

If S is a positive bi-UFS, then

$$\mathcal{U}(S) = \{1\}.$$

Idea of the proof.

- Multiplication by a unit permutes the additive atoms.
- A nontrivial unit u would make the powers $\dots, u^{-1}, 1, u, u^2, \dots$ distinct additive atoms.
- Refining

$$(1 + u + u^2)(1 + u^3) = (1 + u)(1 + u^2 + u^4)$$

produces an impossible finite permutation pattern.

Finding an Irrational Common Atom

Proposition

If S is a positive bi-UFS and $S \neq \mathbb{N}_0$, then there exists an irrational

$$a \in \mathcal{A}_+(S) \cap \mathcal{A}_\times(S).$$

- 1 Choose $s \in S \setminus \mathbb{N}_0$. By BCG, s is irrational.
- 2 Some additive atom in a factorization of s is irrational.
- 3 Factor that additive atom multiplicatively.

Observation

If $\alpha \in \mathcal{A}_+(S)$ and $\alpha = de$, then $d, e \in \mathcal{A}_+(S)$: a nontrivial additive decomposition of d or e would split α by distributivity.

Thus one multiplicative atom divisor is an irrational common atom.

An Additive Obstruction

Proposition

Assume $(S, +)$ is a UFM. If $p \in S^\bullet$ is a nonunit and

$$1 + py = pz$$

for some $y, z \in S^\bullet$, then p is not prime in $S^\bullet$.

The equality gives $p(z - y) = 1$ in $\text{gp}(S, +)$. The additive coordinate cone is then used to construct $r \in S$ such that

$$p \nmid r \quad \text{but} \quad p \mid r^2.$$

This is impossible for a prime element.

Towards a Contradiction

Let $a \in \mathcal{A}_+(S) \cap \mathcal{A}_\times(S)$ be irrational. Start from

$$(1 + a + a^2)(1 + a^3) = (1 + a)(1 + a^2 + a^4).$$

Set

$$P = 1 + a + a^2, \quad Q = 1 + a^3, \quad R = 1 + a, \quad T = 1 + a^2 + a^4.$$

Refinement gives

$$P = uv, \quad Q = wx, \quad R = uw, \quad T = vx.$$

Since $R = 1 + a$ has additive length 2,

$$2 = \ell_+(R) \geq \ell_+(u)\ell_+(w),$$

so only three length pairs are possible:

$$(\ell_+(u), \ell_+(w)) \in \{(1, 2), (2, 1), (1, 1)\}.$$

Case Work

There are three length cases to analyze. Each one produces the same kind of obstruction.

- ❶ If $(\ell_+(u), \ell_+(w)) = (2, 1)$, then $w = 1$ and a multiplicative atom $p \mid (1 + a)$ yields

$$1 + p(ac) = p(cv).$$

- ❷ If $(\ell_+(u), \ell_+(w)) = (1, 1)$, a multiplicative atom divisor of u again yields an equation

$$1 + py = pz.$$

- ❸ If $(\ell_+(u), \ell_+(w)) = (1, 2)$, then $u = 1$ and

$$\frac{1 + a^3}{1 + a} = a^2 - a + 1 \in S.$$

Additive-coordinate comparisons and primality rule out this final case.

Thus every possible refinement leads to a contradiction.

Conclusion of the Bi-UF Proof

The preceding case analysis shows that a positive bi-UFS cannot contain an irrational common atom. Therefore

$$S \neq \mathbb{N}_0 \implies \exists \text{ irrational common atom} \implies \perp.$$

Theorem (Graia, 2026)

A positive semiring S is a bi-UFS if and only if

$$S = \mathbb{N}_0.$$

The Bi-LF Positive Question

Question (Baeth–Chapman–Gotti, 2021)

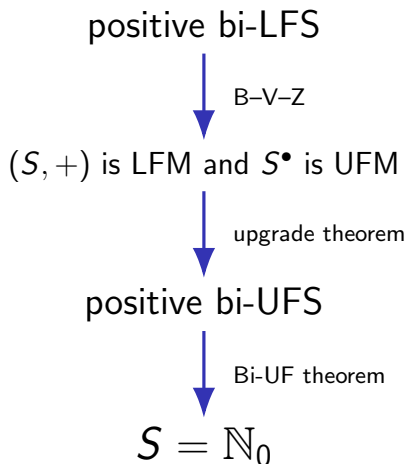
Is $\mathbb{N}_0$ the only positive semiring that is a bi-LFS?

For a positive bi-LFS,

$(S, +)$ is an LFM, $S^\bullet$ is an LFM.

The main issue is whether additive length-factoriality can remain strictly weaker than additive unique factorization when the multiplicative side is factorial.

Proof Architecture for Bi-LF



Only the middle implication requires a new argument.

The Upgrade Theorem

Theorem (Graia–Gotti, 2026)

Let S be a positive semiring. If $(S, +)$ is an LFM and $S^\bullet$ is a UFM, then $(S, +)$ is a UFM.

Proof sketch.

- Assume additive unique factorization fails.
- Length-factoriality forces all additive relations to lie on one line.
- Multiplication by a common atom then forces a mixed relation

$$m + ry = rz.$$

- The coordinate/interval argument shows this relation cannot occur.

Additive Factorization Relations

Let $A = \mathcal{A}_+(S)$. A factorization is encoded by a vector in the free commutative monoid $\mathbb{N}_0^{(A)}$, and

$$\pi : \mathbb{N}_0^{(A)} \longrightarrow S$$

sends the vector to the element it factors.

After group completion, define

$$K = \ker (\mathbb{Z}^{(A)} \longrightarrow \text{gp}(S, +)).$$

Thus K records differences between additive factorizations of the same element.

Example. In $\langle 2, 3 \rangle$,

$$3e_2 - 2e_3 \in K$$

because $3 \cdot 2 = 2 \cdot 3 = 6$.

Length-Factoriality Forces Rank One

Let $\lambda : \mathbb{Z}^{(A)} \rightarrow \mathbb{Z}$ be the length map, obtained by summing the coordinates.

Lemma

If $(S, +)$ is an LFM but not a UFM, then

$$K \cap \ker \lambda = 0, \quad K \neq 0, \quad K = \mathbb{Z}\rho$$

for a primitive relation vector ρ .

Since $\lambda|_K$ is injective, K embeds into $\mathbb{Z}$ and is therefore cyclic.

For $\langle 2, 3 \rangle$, the basic relation is exactly

$$K = \mathbb{Z}(3e_2 - 2e_3).$$

The Interval Argument

The upgrade theorem reduces to ruling out a mixed additive relation

$$m + ry = rz.$$

The goal is to show that this relation is impossible.

Choose a representative $Q \in \mathbb{Z}^{(A)}$ of m/r and consider

$$I = \{t \in \mathbb{R} : Q + t\rho \geq 0\}.$$

The rank-one setup turns the problem into studying the interval I .

- ❶ If I has positive length, rational points with prime denominators force r to divide two coprime ordinary primes.
- ❷ If I is a singleton, the least denominator gives an additive atom b with

$$rb = d.$$

Both possibilities contradict the earlier divisibility restrictions.

Completing the Bi-LF Proof

Assume that $(S, +)$ is an LFM but not a UFM and that $S^\bullet$ is a UFM.

- The rank-one setup gives $K = \mathbb{Z}\rho$.
- Multiplication by an irrational common atom acts on this relation group:

$$B(\rho) = n\rho$$

for some nonzero integer n .

- The lemma rules out the mixed additive factorizations forced by this action. Hence $K = 0$, so $(S, +)$ is a UFM.

Theorem (Graia–Gotti, 2026)

A positive semiring S is a bi-LFS if and only if

$$S = \mathbb{N}_0.$$

- N. R. Baeth, S. T. Chapman, and F. Gotti, *Bi-atomic classes of positive semirings*, Semigroup Forum 103 (2021), 1–23.
- A. Geroldinger and Q. Zhong, *Factorization theory in commutative monoids*, Semigroup Forum 100 (2020), 22–51.
- A. Bu, J. Vulakh, and A. Zhao, *Length-factoriality and pure irreducibility*, Comm. Algebra 51 (2023), 3745–3755.
- Recent Bi-UF progress: Abedi–Arora–Chen–Gotti–Whitford–Wu; Deng–Gotti–Zeng; Gotti–Graia–Han–Liang; Bilakanti–Gotti–Kandasamy–Liang–Liu–Polo–Yang–Yao (2026).

Acknowledgments

I would like to thank

- Dr. Felix Gotti for his guidance, mentorship, and collaboration;
- the SWIM organizers for the opportunity to present this work;
- MIT PRIMES and CrowdMath for providing the research opportunities that made this project possible.

Thank you!